CONNECTEDNESS IN TOPOLOGICAL LINEAR SPACES*

BY

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ABSTRACT

The following properties, well known for normed linear spaces of dimension $\geqq 2$, are established for an arbitrary topological linear space of dimension $\geqq 2$: (a) every neighborhood of 0 contains one whose complement is connected; (b) the complement of a bounded set has exactly one unbounded component.

By a *topological linear space* we mean a real linear space with an *admissible topology*; that is, one with respect to which both vector addition and scalar multiplication are jointly continuous. The T_1 separation axiom is assumed throughout. For each normed linear space of dimension $\geqq 2$, it is obvious that

(a) *every neighborhood of* 0 *contains one whose complement is connected*;

(b) *the complement of a bounded set has exactly one unbounded component.*

These properties play an important role in the study of mappings in normed linear spaces, and in line with efforts to extend the theory Andrzej Granas has asked whether (a) and (b) are valid in an arbitrary topological linear space. We show here that they are. There is an attempt to proceed under minimal hypotheses, so that the results will apply also to certain topologized linear spaces in which the algebraic operations are not jointly continuous.

Henceforth, E will denote a linear space of dimension $\geqq 2$ over the real number field $\mathfrak{R}$, and τ will denote a topology for E. Any assumption about τ's relationship to the algebraic structure of E will be stated explicitly. As is well known [5], each finite-dimensional linear space has a unique admissible topology. This will be called its *natural topology*, and the notion is extended by translation to all finite-dimensional affine subspaces of a linear space.

In treating (a), we make the following assumptions about the topology τ:

(1) There is a neighborhood N of 0 such that $N + N \neq E$.

(2) Every neighborhood of 0 contains a symmetric starshaped neighborhood of 0.

(3) In every (two-dimensional) plane P through 0, the relative topology induced by τ is identical with the natural topology of P.

By an *n-gon* we mean an arc composed of n or fewer line segments.

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THEOREM A. *Suppose that* (E,τ) *is a topologized linear space of dimension* $\geqq 2$ *in which the conditions* (1), (2), *and* (3) *are satisfied. Then every neighborhood* U *of* 0 *contains a neighborhood* W *of* 0 *such that* $E \sim W$ *is connected. Indeed,* W *can be chosen so that each pair of points of* $E \sim W$ *is joined by an* 8*-gon in* $E \sim W$.

Proof. Let N be as in (1) and choose $p \in E \sim (N+N)$. Choose a plane Q which contains the line $\Re p$, and let J be a 2-gon which joins p to $-p$ in Q. By (2), there is a neighborhood V of 0 in E such that $V \cap J = \emptyset$, and by (3) there is a symmetric starshaped neighborhood W of 0 such that $W \subset N \cap U \cap V$. We will show that each point x of $E \sim (W \cup \Re p)$ can be joined to p or to $-p$ by a 3-gon in $E \sim W$, and from this the desired conclusion follows.

Let P be the plane which contains the lines $\Re p$ and $\Re x$, whence $W \cap P$ is a symmetric starshaped subset of P. If the closure $\mathrm{cl}(W \cap P)$ of $W \cap P$ in P does not contain any line through 0, then $W \cap P$ is bounded with respect to the natural topology of P; in this case, x can be joined both to p and to $-p$ by 3-gons in $P \sim W$. Suppose, on the other hand, that the set $\mathrm{cl}(W \cap P)$ does contain a line L through 0. Then there is exactly one such line, for the existence of two would imply that $(W \cap P) + (W \cap P) = P$, whereas $W \subset N$ and $p \in P \sim (N+N)$. Now assume that the points x and p lie on the same side of L in P, and let Z denote the part of P which is bounded by the rays $[0,\infty[x$ and $[0,\infty[p$ and which intersects L only at 0. Then $W \cap Z$ is bounded and starshaped, so x can be joined to p by a 3-gon in $Z \sim W$. Similarly, x can be joined to $-p$ by a 3-gon in $P \sim W$ if x and p lie on opposite sides of L. This completes the proof.

(When E is locally convex, the neighborhood W may be chosen so as to be convex, whence the 8-gons of Theorem A are replaced by 2-gons if W is closed and 3-gons if W is open. These numbers cannot be reduced. When E is locally bounded, W may be chosen so as to be bounded and starshaped, whence the 8-gons are replaced by 3-gons if W is closed and 4-gons if W is open. Can these numbers be reduced? Can the number 8 in Theorem A be reduced for general topological linear spaces?)

In treating (b), we consider a family $\mathscr{B}$ of subsets of E, subject to the following restrictions:

(4) If $X \subset Y$ and $Y \in \mathscr{B}$, then $X \in \mathscr{B}$.

(5) If $Y \in \mathscr{B}$, then $\cup_{y \in Y}[0,y] \in \mathscr{B}$.

(6) If $Y \in \mathscr{B}$, and P is a plane through 0, then with respect to the natural topology of P there is exactly one unbounded component of $P \sim Y$.

THEOREM B. *Suppose that* (E,τ) *is a topologized linear space of dimension* $\geqq 2$, $\mathscr{B}$ *is a family of subsets of* E, *and conditions* (3), (4), (5) *and* (6) *are satisfied. Then for each member* Y *of* $\mathscr{B}$, *exactly one component of* $E \sim Y$ *fails to be a member of* $\mathscr{B}$.

Proof. Let $J = \bigcup_{y \in Y}[0, y]$, whence $J \in \mathscr{B}$ by (5). We claim that for each plane P through 0, the set $P \sim J$ is connected and nonempty. Indeed, $P \sim J$ is a union of rays collinear with 0, where each of the rays is unbounded and connected with respect to the natural topology of P and hence lies in an unbounded component of $P \sim J$. By (6), there is exactly one such component, so $P \sim J$ is connected. Since this is true for every plane P through 0 and since any two points of $E \sim J$ lie together in some such plane, it follows that $E \sim J$ is connected.

Now let U be the component of $E \sim Y$ which contains $E \sim J$, and consider any other component C of $E \sim Y$. Then of course C does not intersect U, whence $C \subset J$ and (by (4)) $C \in \mathscr{B}$. It remains only to show that U is not a member of $\mathscr{B}$. Let $K = \bigcup_{u \in V}[0, u]$ and suppose that $U \in \mathscr{B}$, whence $K \in \mathscr{B}$ by (5). From the definitions of J and K it follows that every ray issuing from 0 is contained in J or in K or in both, and hence one of the sets must contain two such rays R_1 and R_2. Since $R_1 \cup R_2 \in \mathscr{B}$ by (4), it follows from (6) that if P is a plane containing $R_1 \cup R_2$, then $P \sim (R_1 \cup R_2)$ has exactly one unbounded component in the natural topology of P. But this is obviously impossible, and the contradiction completes the proof.

COROLLARY. *Suppose that (E, τ) and $\mathscr{B}$ are as in Theorem B, that the space (E, τ) is locally connected, and that each member of $\mathscr{B}$ is nowhere dense in E. Then $E \sim Y$ is connected whenever* $\mathrm{cl}\, Y \in \mathscr{B}$.

Proof. Since Y is nowhere dense, we have

$$E \sim \mathrm{cl}\, Y \subset E \sim Y \subset \mathrm{cl}(E \sim \mathrm{cl}\, Y).$$

Thus it suffices to show that the set $E \sim \mathrm{cl}\, Y$ is connected. Suppose the contrary, whence from Theorem B it follows that some component T of $E \sim Y$ is a member of $\mathscr{B}$. Since E is locally connected and $E \sim \mathrm{cl}\, Y$ is open, the set T is open and this contradicts the fact that the members of $\mathscr{B}$ are nowhere dense.

Theorems A and B apply to many topologies for linear spaces, and to many choices for the family $\mathscr{B}$. Some of these are quite "pathological." Note, however, that if $\dim E > d \geqq 2$ and if τ is the finest topology for E which agrees with the natural topology on every d-dimensional affine subspace of E, then conditions (1) and (3) are satisfied but (2) is not. (See [4] for discussion of a similar topology.) We do not know whether the conclusion of Theorem A is valid in this case.

In addition to the admissible topologies for a linear space, we consider also the *finite topology*, which is defined as the finest topology agreeing with the natural topology on every finite-dimensional affine subspace. (It appeared under a different name in [1]. The present term was used in [2] and [3].) It is not admissible when $\dim E > \aleph_0$, for then the vector addition is not jointly continuous [3].

Among the many ways in which the members Y of the family $\mathscr{B}$ may be selected we mention the following:

(7) For each neighborhood U of 0, there exists $\mu \in \mathfrak{R}$ such that $\mu U \supset Y$.

(8) The set Y is contained in a linearly bounded convex set.

(9) The intersection of Y with any finite-dimensional affine subspace is bounded in the natural topology.

(10) The closure of Y is compact.

THEOREM C. *Suppose that (E,τ) is a topologized linear space of dimension $\geqq 2$, where τ is an admissible topology or the finite topology. For $i = 7, 8, 9, 10$, let $\mathscr{B}$ denote the family of all subsets Y of E which satisfy the condition* (i). *Then Theorem* A *applies to the space (E,τ) and Theorem* B *applies to the system $(E,\tau,\mathscr{B})$.*

Proof. It is obvious that conditions (1), (3), (4) and (6) are satisfied in each case. For condition (2), consider an arbitrary neighborhood U of 0. By the joint continuity of scalar multiplication (established for the finite topology in [2]), there exist $\varepsilon > 0$ and a neighborhood V of 0 in E such that $]-\varepsilon,\varepsilon[\, V \subset U$. But the set $]-\varepsilon,\varepsilon[\, V$ is a symmetric starshaped neighborhood of 0.

If $\mathscr{B}$ is defined by (8) or (9), the fact that (5) holds is immediate from the relevant definitions. If $\mathscr{B}$ is defined by (7) or (10), it is well known that (5) holds when τ is an admissible topology. Thus to complete the proof, it suffices to establish (5) when $\mathscr{B}$ is defined by (7) or (10) and τ is the finite topology. For this, in turn, it evidently suffices to prove the following:

(*) If Y satisfies condition (7) for the finite topology, then Y is contained in a finite-dimensional linear subspace of E.

In order to prove (*), let us suppose that Y contains an infinite linearly independent set X. Let H be a Hamel basis for E such that $X \subset H$, let $x_1, x_2, \cdots$ be an infinite sequence of distinct members of X, and let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers converging to zero. Finally, let U denote the convex hull of the set

$$(H \sim \{x_1, x_2, \cdots\}) \cup \{\varepsilon_1 x_1, \varepsilon_2 x_2, \cdots\}.$$

Then U is a neighborhood of 0 for the finite topology, and yet Y is not contained in any multiple of U. The contradiction completes the proof of (*) and hence of Theorem C.

Since (7) is the usual notion of boundedness in topological linear spaces, our results supply an affirmative answer to the question of Granas. From the Corollary it follows that if Y is a subset of a topological linear space E, then $E \sim Y$ is connected if any of the following statements is true: E is infinite dimensional and $\mathrm{cl}\, Y$ is compact; E is not locally bounded and Y is bounded; E does not admit a separating family of continuous linear forms and $\mathrm{cl}\, Y$ lies in a linearly bounded convex set.

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